ON AN APPROXIMATE SOLUTION OF CERTAIN PROBLEMS OF SOIL DYNAMICS

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In the propagation through soft soils of pressure waves set up by the incidence on the flat surface of the soil of a shock wave travelling through air, or by an explosion near the surface, the motion of the soil in certain zones is confined to the narrow elongated regions bounded by the surface of the soil and the disturbance front. This enables us to develop a comparatively simple method of solving the problem of the motion of a soil within these regions. The idea behind this method has been borrowed from high-velocity aerodynamics, where it is shown that in the case of a thin body travelling through air at a high supersonic velocity the motion of the air takes place mainly in planes perpendicular to the direction of flight, and the region occupied by the disturbed motion comprises a narrow layer surrounding the body in flight [1,2]. An analysis of the various terms in the equations of gas dynamics shows that if under these circumstances we discard terms of the order of ε^2 (where ε is a small quantity representing the inclination of a generator of the body to the direction of flight), the system of equations can be divided into two parts. The first describes the motion of air in a plane perpendicular to the direction of flight and coincides exactly, after the natural replacement of the axial coordinate by time, with the equations of non-steady plane motion of air. The second part (one equation) describes a disturbance of motion in the direction of flight and can be solved by means of Bernoulli's integral.

A completely analogous state of affairs exists for the soil motion described above. It is found that motion takes place mainly in a direction perpendicular to the surface of the soil, and the whole system of equations of soil dynamics falls approximately into two groups. The first coincides with the exact equations of one-dimensional motion of a soil with plane waves, and describes motion in the direction indicated; it can be solved independently of the second group. The latter describes motion in the two remaining coordinate directions and comprises a system of linear equations the coefficients of which depend on the solution to the first group of equations. The error of an approximate analysis such as this is also of the order of ε^2, where ε is the characteristic value of the angle of inclination of the normal to the disturbance front in the soil to the normal to the surface of the soil.

We would point out that the possibility of applying the principles of high-velocity aerodynamics to the problems of disturbance propagation was established for the case of an ideal liquid by Bagdoev [3].

Consider the non-steady three-dimensional motion of a soil which, however, takes place mainly in one direction, or more precisely, suppose that the components of the velocity vector are of the order

$$u \sim v \sim \varepsilon V_0, \quad w \sim V_0 \tag{1}$$

and that for the derivatives of the various required functions,

$$\frac{\partial}{\partial x} \sim \frac{\partial}{\partial y} \sim \varepsilon \frac{1}{l}, \qquad \frac{\partial}{\partial z} \sim \frac{1}{l}, \qquad \frac{\partial}{\partial t} \sim \frac{1}{t_0} \tag{2}$$

where V_0, l, t_0 are characteristic values of velocity, length and time respectively, and ε is a small parameter.

By means of relations (1) and (2) we can estimate the order of the various terms in the equations of motion of the soil (for these equations we take those given in [4]), and after discarding small quantities, we obtain the required approximate system.

We note that by virtue of (1) and (2) an estimate of the total differential operator $d/dt \equiv \partial/\partial t + v_i\partial/\partial x_i$ is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + w\frac{\partial}{\partial z} + O\left(\varepsilon^2 \frac{V_0}{l}\right) \tag{3}$$

Consequently, for the equation of continuity we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho w)}{\partial z} + O\left(\varepsilon^2 \rho_0 \frac{V_0}{l}\right) = O \tag{4}$$

which, after small quantities of the order of ε^2 are discarded, coincides with the equation of continuity for one-dimensional motion in the direction of the z-axis.

Also, referring to the relations

$$G\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right) = \frac{dS_{ij}}{dt} - S_{ik}\Omega_{jk} - S_{jk}\Omega_{ik} + \lambda S_{ij}$$

$$e_{ij}=\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}, \qquad \Omega_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right) \tag{5}$$

$$\lambda=\frac{2GW-F'(p)\,dp\,/\,dt}{2F(p)}\,e\,[j_2-F(p)]\,e\left[2GW-F'(p)\frac{dp}{dt}\right]$$

$$j_2\equiv\frac{1}{2}S_{ij}S_{ij}, \qquad W\equiv\frac{1}{2}S_{ij}e_{ij}, \qquad S_{ij}=\sigma_{ij}+p\delta_{ij}$$

taken from [4], which relate the components of the stress tensor σ_{ij} to the components of the strain-rate tensor e_{ij}, we obtain with the aid of (1) and (2) the following estimate of the stresses

$$S_{xx}\sim S_{yy}\sim S_{zz}\sim p\sim\sigma_0, \qquad S_{xy}\sim\varepsilon^2\sigma_0, \qquad S_{xz}\sim S_{yz}\sim\varepsilon\sigma_0 \tag{6}$$

where σ_0 is a characteristic value of stress.

Together with the previous estimates these enable us to write down the complete system of approximate equations with an indication of the order of the discarded terms. In the absence of mass forces the equations of motion can be written in the form

$$\rho\left[\frac{\partial u}{\partial t}+w\frac{\partial u}{\partial z}+O\left(\varepsilon^3\frac{V_0^2}{l}\right)\right]=-\frac{\partial p}{\partial x}+\frac{\partial S_{xx}}{\partial x}+\frac{\partial S_{xz}}{\partial z}+O\left(\varepsilon^3\frac{\sigma_0}{l}\right)$$

$$\rho\left[\frac{\partial v}{\partial t}+w\frac{\partial v}{\partial z}+O\left(\varepsilon^3\frac{V_0^2}{l}\right)\right]=-\frac{\partial p}{\partial y}+\frac{\partial S_{yy}}{\partial y}+\frac{\partial S_{yz}}{\partial z}+O\left(\varepsilon^3\frac{\sigma_0}{l}\right)$$

$$\rho\left[\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}+O\left(\varepsilon^2\frac{V_0^2}{l}\right)\right]=-\frac{\partial p}{\partial z}+\frac{\partial S_{zz}}{\partial z}+O\left(\varepsilon^2\frac{\sigma_0}{l}\right) \tag{7}$$

Relations (5) now become

$$G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\frac{\partial S_{xy}}{\partial t}+w\frac{\partial S_{xy}}{\partial z}-(S_{xx}-S_{yy})\,\Omega_{yx}-S_{xz}\,\Omega_{yz}-$$
$$-S_{yz}\,\Omega_{xz}+\lambda S_{xy}+O\left(\varepsilon^4\sigma_0\frac{V_0}{l}\right)$$

$$G\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=\frac{\partial S_{xz}}{\partial t}+w\frac{\partial S_{xz}}{\partial z}-(S_{xx}-S_{zz})\,\Omega_{zx}+\lambda S_{xz}+O\left(\varepsilon^3\sigma_0\frac{V_0}{l}\right)$$

$$G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=\frac{\partial S_{yz}}{\partial t}+w\frac{\partial S_{yz}}{\partial z}-(S_{yy}-S_{zz})\,\Omega_{zy}+\lambda S_{yz}+O\left(\varepsilon^3\sigma_0\frac{V_0}{l}\right)$$

$$2G\left[-\frac{1}{3}\frac{\partial w}{\partial z}+O\left(\varepsilon^2\frac{V_0}{l}\right)\right]=\frac{\partial S_{xx}}{\partial t}+w\frac{\partial S_{xx}}{\partial z}+\lambda S_{xx}+O\left(\varepsilon^2\sigma_0\frac{V_0}{l}\right)$$

$$2G\left[-\frac{1}{3}\frac{\partial w}{\partial z}+O\left(\varepsilon^2\frac{V_0}{l}\right)\right]=\frac{\partial S_{yy}}{\partial t}+w\frac{\partial S_{yy}}{\partial z}+\lambda S_{yy}+O\left(\varepsilon^2\sigma_0\frac{V_0}{l}\right)$$

$$2G\left[\frac{2}{3}\frac{\partial w}{\partial z}+O\left(\varepsilon^2\frac{V_0}{l}\right)\right]=\frac{\partial S_{zz}}{\partial t}+w\frac{\partial S_{zz}}{\partial z}+\lambda S_{zz}+O\left(\varepsilon^2\sigma_0\frac{V_0}{l}\right) \tag{8}$$

where the expressions for W and dp/dt in the formula for λ should be replaced by

$$W = S_{zz}\frac{\partial w}{\partial z} + O\left(\varepsilon^2\sigma_0\frac{V_0}{l}\right), \qquad \frac{dp}{dt} = \frac{\partial p}{\partial t} + w\frac{\partial p}{\partial z} + O\left(\varepsilon^2\sigma_0\frac{V_0}{l}\right) \tag{9}$$

We add to Equations (7) and (8) the relations

$$p = f(\rho, \rho_*), \qquad \frac{d\rho_*}{dt} = \frac{d\rho}{dt} e(\rho - \rho_*) e\left(\frac{d\rho}{dt}\right) \tag{10}$$

Here d/dt has the meaning given in (3). Equations (7), (8) and (10) form the complete system. It will be seen that this system of approximate equations can be divided into a system of nonlinear equations of one-dimensional motion along the z-axis (Equation (4), the last of Equations (7), the last three of Equations (8) and Equations (10)) and a system of equations (the remainder) which describes motion parallel to the xy-plane. This second group of equations can be further divided into a linear system, which determines u, v, S_{xz} and S_{yz}, and a separate equation, which determines S_{xy} and which is also linear.

We shall consider now the conditions under which it is possible to solve a problem with the aid of the above equations. It follows from (2) that it is possible to do so on condition that the ratio of the characteristic diemsnion l_z in the direction of the z-axis of the region of the soil undergoing motion to the characteristic dimension l_{xy} in directions perpendicular to the z-axis is small, i.e.

$$l_z \sim \varepsilon l_{xy} \tag{11}$$

If motion of the soil is generated by the action of a shock-wave travelling in air over the surface of the soil with a velocity of the order of D, then $l_{xy} \sim Dt_0$. The characteristic dimension of the region in the z direction will be of the order $l_z \sim at_0$, where a is the characteristic value of the wave velocity (shock or acoustic) in the soil. Therefore condition (11) reduces to

$$a \sim \varepsilon D \ll D \tag{12}$$

Further consideration of the compatibility conditions on the front of a shock wave travelling in a soil at an inclination of the order of ε, as given by (12), leads to the conclusion that the estimates provided by (1) will be valid on this front. Making the reasonable assumption that this is sufficient to prove their validity everywhere within the region of motion, we conclude that (12) is a solution to the question of the conditions under which the proposed approximate method can be employed.

In an actual application of this method the following procedure would be adopted.

Knowing the pressure distribution behind the airborne shock wave

$$-\sigma_z(x, y, 0, t) = p_0 f(x / l_{xy}, y / l_{xy}, t / t_0) \tag{13}$$

we would have for every column of soil in the direction of the z-axis with coordinates x and y the law of variation with time of the pressure acting on the base of the column. This is sufficient for us to solve the one-dimensional problem of the motion of such a column. This solution, by virtue of (13), will depend on x and y as parameters. Of course, in deriving this solution we have to solve a finite number of one-dimensional problems for a finite choice of values of x and y, and for intermediate values interpolation is used. Thus we would have found the functions w, p, ρ, ρ_*, S_{xx}, S_{yy} and S_{zz}. We now solve the corresponding linear problem, and as a result determine the values of u, v, S_{xz}, S_{yz} and S_{xy}. In this way the problem of the three-dimensional motion of the soil would be solved approximately. The error would be of the order of $\varepsilon^2 \sim (a/D)^2$. Consequently, even if $\varepsilon \sim 0.3$, the error would be only 10%, which for many cases is quite acceptable.

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